Associate and conjugate minimal immersions in $\mathbb{M} \times \mathbb{R}$

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1. Introduction

A beautiful phenomenon in Euclidean space is the existence of a 1-parameter family of minimal isometric surfaces connecting the catenoid and the helicoid. They are *associate*. A well-known fact, is that any two conformal isometric minimal surfaces in a space form are associate. What happens in other 3-dimensional manifolds ?

In this paper we will discuss the same phenomenon in the product space, $\mathbb{M} \times \mathbb{R}$, giving a definition of associate minimal immersions. We specialize in the situations $\mathbb{M} = \mathbb{H}^2$, the hyperbolic plane, and $\mathbb{M} = \mathbb{S}^2$, the sphere where surprising facts occur. We will prove some existence and uniqueness results explained in the sequel. We begin with the definition.

Let $\mathbb{M}$ be a two dimensional Riemannian manifold. Let (x, y, t) be local coordinates in $\mathbb{M} \times \mathbb{R}$, where $z = x + iy$ are conformal coordinates on $\mathbb{M}$ and $t \in \mathbb{R}$. Let $\sigma^2 |\,\mathrm{d}z|^2$, be the conformal metric in $\mathbb{M}$, hence $\mathrm{d}s^2 = \sigma^2 |\,\mathrm{d}z|^2 + \mathrm{d}t^2$ is the metric in the product space $\mathbb{M}\times\mathbb{R}$. Let $\Omega \subset \mathbb{C}$ be a planar simply connected domain, $w = u + iv \in \Omega$. We recall that if $X : \Omega \to \mathbb{M} \times \mathbb{R}$, $w \mapsto (h(w), f(w))$, $w \in \Omega$, is a conformal minimal immersion then $h : \Omega \to (\mathbb{M}, \sigma^2 |\,\mathrm{d}z|^2)$ is a harmonic map. We recall also that for any harmonic map $h : \Omega \subset \mathbb{C} \to \mathbb{M}$ there exists a related Hopf holomorphic function $Q(h)$. Two conformal immersions $X = (h, f),\ X^* = (h^*, f^*) : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ are said *associate* if they are isometric and if the Hopf functions satisfy the relation $Q(h^*) = e^{2i\theta} Q(h)$ for a real number θ. If $Q(h^*) = -Q(h)$ then the two immersions are said *conjugate*.

In this paper we will show that there exist two conformal isometric minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$, with constant *Gaussian curvature* -1, that are non associate. We will prove also that the vertical cylinder over a planar geodesic in $\mathbb{H}^2 \times \mathbb{R}$, are the only minimal surfaces with constant *Gaussian curvature* $K \equiv 0$.

One of our principal results is a uniqueness theorem in $\mathbb{H}^2 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}$, showing that the conformal metric and the Hopf function determine a minimal conformal immersion, up to an isometry of ambient space, see Theorem 4. We will derive the existence of the minimal associate family in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

Date: February 2, 2008.

The authors likes to thanks CNPq and PRONEX of Brazil and Accord Brasil-France, for partial financial support.

in corollary 8 by establishing an existence result, see Theorem 5. The associate minimal family in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ is derived by another approach in [3].

The first author has constructed examples of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and in $\mathbb{S}^2 \times \mathbb{R}$ which generalize the family of Riemann's minimal examples of $\mathbb{R}^3$. He classify and construct all example foliated by horizontal constant curvature curves. Some of them have Gaussian curvature $K \equiv -1$. This family is parametrized by two parameter (c, d) and the example corresponding to (c, d) is conjugate to the one parametrized by (d, c) (we refer to the paper [7] for more details on these surfaces). The second and third authors, proved that any two minimal isometric screw motion immersions in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ are associate, see [14]. The second author proved that any two minimal isometric parabolic screw motion immersions into $\mathbb{H}^2 \times \mathbb{R}$ are associate. On the other hand, he proved that there exist families of *associate hyperbolic* screw motion immersions, but there exist also isometric non-associate *hyperbolic* screw motion immersions, see [12]. There exists hyperbolic screw motion surfaces associate to parabolic screw motion surfaces (see example 13).

Several questions arise from this work: We point out the problem of the existence of the associate minimal family in $\mathbb{M} \times \mathbb{R}$, for any 2-dimensional Riemannian manifold $\mathbb{M}$. Also we may ask in which general assumptions isometric immersions must be associate ?

The second principal result is a generalization of the Krust's theorem (see [6], tome I, page 118 and applications therein) which state that an associate surface of a minimal vertical graph on a convex domain is a vertical graph. These theorem is true in $\mathbb{M} \times \mathbb{R}$ when the Gaussian curvature $K_{\mathbb{M}} \leq 0$. It will not be true anymore by example in $\mathbb{S}^2 \times \mathbb{R}$ (see Theorem 12).

For related works on minimal surfaces in $\mathbb{M} \times \mathbb{R}$, see for instance Daniel [3] Nelli and Rosenberg [10], Meeks and Rosenberg [9] and Rosenberg [11].

2. Preliminar

We consider $X : \Omega \subset \mathbb{R}^2 \to \mathbb{M} \times \mathbb{R}$ a minimal surface conformaly embedded in a product space. $\mathbb{M}$ is a Riemannian complete two-manifold with metric $\mu = \sigma^2(z)|dz|^2$ and Gauss curvature $K_{\mathbb{M}}$. First we fix some notations. Let us denote $|v|_\sigma^2 = \sigma^2|v|$, $\langle v_1, v_2\rangle_\sigma = \sigma^2\langle v_1, v_2\rangle$ where $|v|$ and $< v_1, v_2 >$ stands for the standard norm and inner product in $\mathbb{R}^2$. Let us find $w = u + iv$ as conformal parameters of Ω, i.e. $ds_X^2 = \lambda^2|\,\mathrm{d}w|^2$. We denote by $X = (h, f)$ the immersion where $h(w) \in \mathbb{M}$ and $f(w) \in \mathbb{R}$.

Assume that $\mathbb{M}$ is isometrically embedded in $\mathbb{R}^k$. By definition (see Lawson [5]) the mean curvature vector in is

$$2\overrightarrow{H} = (\triangle X)^{T_X\mathbb{M}\times\mathbb{R}} = ((\triangle h)^{T_h\mathbb{M}}, \triangle f) = 0$$

where $h = (h_1, ..., h_k)$. Then $h : \Omega \longrightarrow \mathbb{M}$ is a harmonic map between Ω and the complete Riemannian surface $\mathbb{M}$ and f is a real harmonic function. If $(U, \sigma^2(z)|\,\mathrm{d}z|^2)$ is a local parametrization of $\mathbb{M}$, the harmonic map equation in the complex coordinate $z = x + iy$ of $\mathbb{M}$ (see [13], page 8) is

$$h_{w\bar{w}} + 2(\log\sigma\circ h)_z h_w h_{\bar{w}} = 0 \tag{1}$$

In the theory of *harmonic map* there is two important classical object to consider. One is the holomorphic *quadratic Hopf differential* associate to h:

$$Q(h) = (\sigma\circ h)^2 h_w \overline{h}_w (\mathrm{d}w)^2 := \phi(w)(dw)^2 \tag{2}$$

An other object is the *complex coefficient of dilatation* (see Alhfors [2]) of a quasi-conformal map:

$$a(w) = \frac{\overline{h_{\bar{w}}}}{h_w}$$

Since we consider conformal immersion, we have $(f_w)^2 = -\phi(w)$ from (see [14]):

$$|h_u|^2_\sigma + (f_u)^2 = |h_v|^2_\sigma + (f_v)^2$$

$$\langle h_u, h_v\rangle_\sigma + f_u.f_v = 0$$

We define η as the holomorphic one form $\eta = \pm 2i\sqrt{\phi(w)}\,\mathrm{d}w$ when ϕ have only even zeroes. The sign is chosen in function of f to have:

$$f = \mathrm{Re}\int_w \eta \tag{3}$$

In the case where $\mathbb{M} = \mathbb{R}^2$, g is the Gauss map in the classical Weierstrass representation.

When X is a conformal immersion then the Gauss map N in $\mathbb{M}^2\times\mathbb{R}$ is given by (see [14]):

$$N = \frac{(\frac{2}{\sigma}\mathrm{Re}g, \frac{2}{\sigma}\mathrm{Im}g, |g|^2 - 1)}{|g|^2 + 1} \tag{4}$$

where $g := \dfrac{f_w h_{\overline{w}} - f_{\overline{w}} h_w}{\sigma|h_{\overline{w}}|(|h_w| + |h_{\overline{w}}|)}$. We remark that

$$g^2 = -\frac{h_w}{\overline{h_{\bar{w}}}} \tag{5}$$

Using the equations above (2),(5) we can express the differential dh as follows:

$$dh = h_{\bar{w}} d\bar{w} + h_w dw = \frac{1}{2\sigma}\overline{g^{-1}\eta} - \frac{1}{2\sigma} g\eta \tag{6}$$

The metric $ds_X^2 = \lambda^2 |\,\mathrm{d}w|^2$ is given by [14]:

$$ds_X^2 = (|h_w|_\sigma + |h_{\bar{w}}|_\sigma)^2 |\,\mathrm{d}w|^2 \tag{7}$$

Thus combining together the equation we derive the metric in terms of g and η:

$$ds_X^2 = \frac{1}{4}(|g|^{-1} + |g|)^2 |\eta|^2 = (|\sqrt{a}| + |\sqrt{a}|^{-1})^2 |\phi||dw|^2 \tag{8}$$

In the case of minimal surfaces X conformally immersed in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, the data (g, η) are classical Weierstrass data:

$$X(w) = (h, f) = \left(\frac{1}{2}\overline{\int_w g^{-1}\eta} - \frac{1}{2}\int_w g\eta, \mathrm{Re}\int_w \eta \right)$$

The main difference is in the fact that g is no more a meromorphic map in the case where the ambient space is $\mathbb{M} \times \mathbb{R}$. To study g it is more convenient for some purpose to consider the complex function $\omega + i\psi$ defined by

$$g := -ie^{\omega + i\psi} \tag{9}$$

We will derive some equation for ψ in Lemma 9 in term of ω. It is a well known fact (see [13] page 9) that harmonic mappings satisfy the Böchner formula:

$$\triangle_0 \log \frac{|h_w|}{|h_{\bar{w}}|} = -2K_{\mathbb{M}} J(h) \tag{10}$$

where $J(h) = \sigma^2 \left(|h_w|^2 - |h_{\bar{w}}|^2 \right)$ is the Jacobian of h with $|h_w|^2 = h_w \overline{h_w}$. Hence taking into account (2), (5), (9) and (10):

$$\triangle_0 \omega = -2K_{\mathbb{M}} \sinh(2\omega)|\phi| \tag{11}$$

where $\triangle_0$ denote the laplacian in the euclidean metric. With these convention notice that the metric and the third coordinate of the Gauss map N are given by

$$ds_X^2 = 4\cosh^2 \omega |\phi||\,\mathrm{d}w|^2 \quad \text{and} \quad N_3 = \tanh \omega$$

On account of the above discussion we deduce the following:

Proposition 1. *Let $h : \Omega \to \mathbb{M}$ be a harmonic mapping such that the holomorphic quadratic differential $Q(h)$ does not vanish or so have zero with even order. Then there exists a complex map $g = -ie^{\omega+i\psi}$ and a holomorphic one form $\eta = \pm 2i\sqrt{Q}(h)$ such that, with $f = \mathrm{Re}\int \eta$, the map $X := (h, f) : \Omega \to \mathbb{M}\times\mathbb{R}$ is a conformal and minimal (possibly branched) immersion. The third component of the normal vector is given by $N_3 = \tanh\omega$. The metric of the immersion is given by (8):*

$$ds_X^2 = \cosh^2\omega|\eta|^2$$

where ω is a solution of the sh-Gordon equation

$$\triangle_0\omega = -2K_{\mathbb{M}}\sinh(2\omega)|\phi|.$$

Proof

We deduce from the hypothesis that we can solve in f the equation $(f_w)^2 = -(\sigma\circ h)^2 h_w\overline{h}_w$ (since Ω is simply connected). Therefore the real function f is harmonic and the map $X := (h, f) : \Omega \to \mathbb{M}\times\mathbb{R}$ is a conformal and minimal (possibly branched) immersion. Observe that $X^* := (h, -f)$ also defines a conformal and minimal (possibly branched) immersion into $\mathbb{M}\times\mathbb{R}$, isometric to X with $g^* = -g$ and $\eta^* = -\eta$. $\square$

We denote by $\mathbb{R}^{2,1}$ the Minkowski 3-space, that is $\mathbb{R}^3$ equipped with the Lorentzian metric $\overline{\nu} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$ where (x_1, x_2, x_3) are the coordinates in $\mathbb{R}^3$. We consider the hyperboloid $\mathcal{H}$ in $\mathbb{R}^{2,1}$ defined by

$$\mathcal{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1},\ x_1^2 + x_2^2 - x_3^2 = -1\},$$

$\mathcal{H}$ has two connected components, we call $\mathcal{H}_+$ the component for which $x_3 \geq 1$, we call $\mathcal{H}_-$ the other component. It is well-known that the restriction of $\overline{\nu}$ to $\mathcal{H}_+$ is a regular metric ν_+ and that $(\mathcal{H}_+, \nu_+)$ is isometric to the hyperbolic plane $\mathbb{H}^2$. We define in the same way the metric ν_- on $\mathcal{H}_-$ and $(\mathcal{H}_-, \nu_-)$ is also isometric to the hyperbolic plane $\mathbb{H}^2$. Throughout this paper we always choose as model for $\mathbb{H}^2$ the unit disc $\mathbb{D}$ equipped with the metric $\sigma^2|dz|^2 = \frac{4}{(1-|z|^2)^2}|dz|^2$. The isometries $\Pi_+ : \mathcal{H}_+ \to \mathbb{D}$ and $\Pi_- : \mathcal{H}_- \to \mathbb{D}$ are given by

$$\Pi_+(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 + x_3},\quad \forall (x_1, x_2, x_3) \in \mathcal{H}_+$$

$$\Pi_-(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3},\quad \forall (x_1, x_2, x_3) \in \mathcal{H}_-$$

In fact Π_+ (resp. Π_-) is the stereographic projection from the south pole $(0, 0, -1) \in \mathcal{H}_-$ (resp. from the north pole $(0, 0, 1) \in \mathcal{H}_+$). Observe that in [1],

keeping the notations, ψ_2 is the conjugate of the stereographic projection from the south pole, that is $\psi_2 = \overline{\Pi_+}$.

Let $\Omega \subset \mathbb{C}$ be a connected and simply connected open subset with $w = u + iv$ the coordinates on Ω. An immersion $X : \Omega \to \mathbb{R}^{2,1}$ is said to be a *spacelike immersion* if for every point $p \in \Omega$ the restriction of $\overline{\nu}$ at the tangent space $T_pX(\Omega)$ is a regular metric. In this work we only consider spacelike immersions. Let $X : \Omega \to \mathbb{R}^{2,1}$ be a spacelike immersion. For each $p \in \Omega$ there is a unique vector $N(p) \in \mathcal{H}$ such that $(X_u, X_v, N)(p)$ is a positively oriented basis and $N(p)$ is orthogonal to $X_u(p)$ and $X_v(p)$. This defines a map $N : \Omega \to \mathcal{H}$ called the *Gauss map*.

We will make now some comments about an existence theorem of spacelike mean curvature one surfaces in Minkowski three space and their relation with harmonic maps, inferred by Akutagawa and Nishikawa [1].

Let $h : \Omega \to \mathbb{D}$ be a harmonic map, that is h satisfies (1):

$$h_{w\overline{w}} + \frac{2\overline{h}}{1 - |h|^2} h_w h_{\overline{w}} = 0.$$

We assume that neither h nor $\overline{h}$ is holomorphic. It is shown in [1] Theorem 6.1, that given such h there exists an (possibly branched) immersion $X_+ : \Omega \to \mathbb{R}^{2,1}$ such that the Gauss map is $N_+ = \Pi_+^{-1} \circ h$, furthermore the mean curvature is constant and equals to 1 and the induced metric on Ω is

$$\nu_+ = \frac{4|h_w|^2}{(1 - |h|^2)^2} |\,\mathrm{d}w|^2.$$

We give the correspondence between our notations and the notations of [1]: $N_+ = G$ and $h = \overline{\Psi_2}$. The map X_+ is unique up to a translation. In the same way there exists an unique up to a translation (possibly branched) immersion $X_- : \Omega \to \mathbb{R}^{2,1}$ such that the Gauss map is $N_- = \Pi_-^{-1} \circ h$, furthermore the mean curvature is constant and equals to 1 and the induced metric on Ω is

$$\nu_- = \frac{4|\overline{h}_w|^2}{(1 - |h|^2)^2} |\,\mathrm{d}w|^2,$$

with $N_- = G$ and $h = \Psi_1$ Let us note that those two (branched) immersions are not isometric and that the Gauss map of X_+ (resp. X_-) takes values in $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). In this paper we are only concerned with the immersion X_+.

On account of the above discussion we deduce:

Lemma 2. *Let $X = (h, f) : \Omega \to \mathbb{M} \times \mathbb{R}$ be a conformal immersion. Let $N = (N_1, N_2, N_3)$ be the Gauss map of X. Let K (resp. K_{ext}) be the intrinsic*

(resp. extrinsic) curvature of X. At last we denote by $K_{\mathbb{M}}$ the Gauss curvature of $\mathbb{M}$*. Then the* Gauss equation *of X reads as*

$$(\textit{Gauss Equation}) \qquad K(w) - K_{ext}(X(w)) = K_{\mathbb{M}}(h(w))N_3^2(w),$$

for each $w \in \Omega$.

Proof

As usual $z = x + iy$ is a local and conformal coordinate of $\mathbb{M}$ and t is the coordinate on $\mathbb{R}$. We denote by $\overline{R}$ the tensor of curvature of $\mathbb{M} \times \mathbb{R}$, that is

$$\overline{R}(A,B)C = \overline{\nabla}_A\overline{\nabla}_B C - \overline{\nabla}_B\overline{\nabla}_A C - \overline{\nabla}_{[A,B]}C,$$

for any vector fields A, B, C on $\mathbb{M} \times \mathbb{R}$ where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{M} \times \mathbb{R}$.

As X is a conformal immersion the induced metric on Ω has the form $ds_X^2 = \lambda^2(w)|\,\mathrm{d}w^2|$ with $\lambda = (\sigma \circ h)(|h_w| + |\overline{h}_w|)$. The Gauss equation is

$$K(w) - K_{ext}(X(w)) = \frac{\langle \overline{R}(X_u, X_v)X_u\ ; X_v\rangle}{\lambda^4}(w),$$

where $\langle\ ;\ \rangle$ is the scalar product on $\mathbb{M} \times \mathbb{R}$, $X_u = \frac{\partial X}{\partial u} = (\mathrm{Re}h)_u\partial_x + (\mathrm{Im}h)_u\partial_y + f_u\partial_t$ and so on. A tedious but straightforward computation shows that

$$\begin{aligned}
\overline{R}(\partial_x, \partial_y)\partial_x &= \ -\Delta\log(\sigma)\partial_y \\
\overline{R}(\partial_x, \partial_y)\partial_y &= \Delta\log(\sigma)\partial_x \\
\overline{R}(\partial_x, \partial_x)\partial_* &= \overline{R}(\partial_y, \partial_y)\partial_* = 0 \\
\overline{R}(\partial_t, \partial_*)\partial_* &= \overline{R}(\partial_*, \partial_t)\partial_* = \overline{R}(\partial_*, \partial_*)\partial_t = 0,
\end{aligned}$$

where ∂_* stands for any vector field among ∂_x, ∂_y or ∂_t and Δ is the euclidean Laplacian. We deduce that

$$\langle \overline{R}(X_u, X_v)X_u; X_v\rangle = -\sigma^2\Delta\log(\sigma)(|h_w|^2 - |h_{\overline{w}}|^2)^2.$$

Let us observe that $K_{\mathbb{M}} = -\Delta\log(\sigma)/\sigma^2$ therefore we deduce from equation (4) that $N_3 = (|h_w| - |h_{\overline{w}}|)/(|h_w| + |h_{\overline{w}}|)$. Now using the expression of λ we get the result, which concludes the proof. $\square$

Notice that given a geodesic $\Gamma \subset \mathbb{H}^2 \times \{0\}$, the vertical cylinder $\mathcal{C}$ over Γ defined by $\mathcal{C} := \{(x, y, t);\ x + iy,\ t \in \mathbb{R}\} \subset \mathbb{H}^2 \times \mathbb{R}$ is a minimal surface with Gauss curvature $K \equiv 0$. We now deduce the following.

***Corollary* 3.** *let $X = (h, f) : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and minimal immersion. Let $w \in \Omega$ be such that $K(w) = 0$ where K stands for the Gauss curvature of X (that is the intrinsic curvature).*

Then the tangent plane of $X(\Omega)$ at $X(w)$ is vertical. Therefore if $K \equiv 0$ then $X(\Omega)$ is part of a vertical cylinder over a planar geodesic plane of $\mathbb{H}^2 \times \mathbb{R}$, that is, there exists a geodesic Γ of $\mathbb{H}^2 \times \{t\}$ such that $X(\Omega) \subset \Gamma \times R$.

Proof

As $\mathbb{M} = \mathbb{H}^2$ we have $K_{\mathbb{M}} \equiv -1$. Using the Gauss equation, see the lemma 2, we deduce that if $K(w) = 0$ at some point $w \in \Omega$ then

$$(*) \qquad K_{ext}(X(w)) = N_3^2(w).$$

Recall that the extrinsic curvature K_{ext} is the ratio between the determinants of the second and the first fundamental forms of X. Therefore as X is a minimal immersion we have $K_{ext}(X(w)) \leq 0$ at any point w. Using $(*)$ we obtain that $N_3^2(w) = 0$, that is the tangent plane is vertical at $X(w)$.

Furthermore, if $K \equiv 0$ we deduce that at each point the tangent plane is vertical. Using this fact we get that at any point $X(w)$ the intersection of $X(\Omega)$ with the vertical plane at $X(w)$ spanned by $N(w)$ and ∂_t is part of a vertical straight line. We deduce that there exists a planar curve $\Gamma \subset \mathbb{H}^2 \times \{0\}$ such that $X(\Omega) \subset \Gamma \times \mathbb{R}$. Again, as X is minimal we obtain that the curvature of Γ always vanishes, that is Γ is a geodesic of $\mathbb{H}^2$. $\square$

3. Minimal immersions in $\mathbb{M} \times \mathbb{R}$

Next we suppose that $\mathbb{M} = \mathbb{R}^2, \mathbb{H}^2$ or $\mathbb{S}^2$. In case where $\mathbb{M} = \mathbb{R}^2$ we have $\sigma(z) \equiv 1$, if $\mathbb{M} = \mathbb{H}^2$ we consider the model of the unit disk $\mathbb{D}$ and then $\sigma(z) = 2/(1 - |z|^2)$ for every $z \in \mathbb{D}$. At last if $\mathbb{M} = \mathbb{S}^2$ we can choose among the coordinate charts $\mathbb{R}^2$ given by the stereographic projections with respect to the north pole and the south pole, we have in both cases $\sigma(z) = 2/(1 + |z|^2)$ for every $z \in \mathbb{R}^2$.

Theorem 4. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and consider two isometric and conformal minimal immersions $X, X^* : \Omega \to \mathbb{M} \times \mathbb{R}$. Let us call h (resp. h^*) the horizontal component of X (resp. X^*). Assume that h and h^* share the same Hopf quadratic differential.*

Then X and X^* are equal up to an isometry of $\mathbb{M} \times \mathbb{R}$.

Proof

Let us set $X = (h, f)$ where $h : \Omega \to \mathbb{M}$ is the horizontal component and $f : \Omega \to \mathbb{R}$ is the vertical component. In the same way let us set $X^* = (h^*, f^*)$. We will use the map $g = -ie^{\omega + i\psi}$ (resp. $g^* = -ie^{\omega^* + i\psi^*}$) associated to h (resp. h^*) defined in the introduction and the one form η (resp. η^*). As X and X^* are isometric immersions we infer from (8):

$$\frac{1}{4}(|g|+|g|^{-1})^2|\eta| = \frac{1}{4}(|g^*|+|g^*|^{-1})^2|\eta^*|$$

Also as h and h^* share the same Hopf quadratic differential $Q = \phi dw^2$ we have

$$|\eta| = 2|\phi|^{1/2}| = |\eta^*|$$

We deduce that we have

$$(*) \qquad |g| = |g^*|,$$

or

$$(**) \qquad |g| = |g^*|^{-1},$$

If case $(**)$ happens we consider the new immersion $X^{**} : \Omega \to \mathbb{M}\times\mathbb{R}$ defined by $X^{**} = (\overline{h^*}, f^*)$. Now case $(*)$ happens considering immersion X^{**}, with datas $g^{**} = (g^*)^{-1}$ and $\eta^{**} = \eta^*$. Note that X^{**} and X are isometric immersions with same Hopf quadratic differential. Therefore, up to an isometry of $\mathbb{M}\times\mathbb{R}$, we can assume that case $(*)$ happens and $\omega = \omega^*$.

Let us assume now that $\mathbb{M} = \mathbb{H}^2$, the case $\mathbb{M} = \mathbb{S}^2$ is similar and case $\mathbb{M} = \mathbb{R}^2$ will be considered later.

Let us consider the Minkowski 3-space $\mathbb{R}^{2,1}$. As $h : \Omega \to \mathbb{H}^2$ is a harmonic map and Ω is simply connected it is known that there exists a CMC one (possibly branched) immersion $\widetilde{X} : \Omega \to \mathbb{R}^{2,1}$ such that the Gauss map is $\Pi_+^{-1} \circ h$. Furthermore the induced metric on Ω is given by

$$ds^2_{\widetilde{X}} = ((\sigma\circ h)|h_w|)^2 \,|\,\mathrm{d}w|^2 = e^{2\omega}|\phi||\,\mathrm{d}w|^2,$$

see the Section 2 (Preliminar). Notice that ϕ can vanish only at isolated points, so there exists a simply connected open subset V of Ω, $V \subset \Omega$, such that $\widetilde{X}$ defines a regular immersion from V into $\mathbb{R}^{2,1}$ and $ds^2_{\widetilde{X}}$ defines a regular metric.

Furthermore we deduce from Theorem 3.4 of [1] that the second fundamental form of $\widetilde{X}$ is given uniquely in term of Q and $ds^2_{\widetilde{X}}$. To see this, observe first that, setting $\tilde{\phi}(\widetilde{X}) := \frac{1}{2}(b_{uu} - b_{vv} - i2b_{uv})$, we get from relation (3.12) of [1] that

$$\tilde{\phi}(\widetilde{X}) = (\sigma\circ h)^2 h_w \overline{h}_w = \phi, \tag{12}$$

that is $\tilde{\phi}(\widetilde{X})\,\mathrm{d}w^2 = Q(h)$. Pay attention to the fact that our conventions are not the same as in [1], for example following notation of [1] we have $b_{uu} = ((\sigma\circ h)|h_w|)^2\, h_{11}$ and so on, therefore $\tilde{\phi}(\widetilde{X}) = ((\sigma\circ h)|h_w|)^2\,\phi$ where ϕ is given in [1] that is $\phi = \frac{1}{2}(h_{11} - h_{22} - i2h_{12})$.

Now using the fact that $b_{uu}+b_{vv}=2\left((\sigma\circ h)|h_w|\right)^2$ (since the mean curvature is 1), we deduce that

$$
\begin{aligned}
b_{uu}(\widetilde{X}) &= ((\sigma\circ h)|h_w|)^2+\mathrm{Re}\,Q(h)/\,\mathrm{d}w^2 = e^{2\omega}|\phi|+\mathrm{Re}\phi && (13)\\
b_{vv}(\widetilde{X}) &= ((\sigma\circ h)|h_w|)^2-\mathrm{Re}\,Q(h)/\,\mathrm{d}w^2 = e^{2\omega}|\phi|-\mathrm{Re}\phi && (14)\\
b_{uv}(\widetilde{X}) &= -\mathrm{Im}\,Q(h)/\,\mathrm{d}w^2 = -\mathrm{Im}\phi && (15)
\end{aligned}
$$

In the same way there exists an unique (up to a translation) CMC one (possibly branched) immersion $\widetilde{X^*}:\Omega\to\mathbb{R}^{2,1}$ such that the Gauss map is $\Pi_+^{-1}\circ h^*$. We can assume that $\widetilde{X^*}$ defines a regular immersion on V. Notice that we have $Q(h)=Q(h^*)$ and identities $(*)$ as well. We deduce from the former discussion that $\widetilde{X}$ and $\widetilde{X^*}$ share the same induced metric on V and the same second fundamental form. Therefore we infer with the fundamental theorem of geometry in Minkowski 3-space that $\widetilde{X}$ and $\widetilde{X^*}$ are equal up to a positive isometry Γ in $\mathbb{R}^{2,1}$, that is $\widetilde{X^*}=\Gamma\circ\widetilde{X}$. The restriction of Γ on $\mathbb{H}^2$ defines an isometry γ of $\mathbb{H}^2$ and we get $h^*=\gamma\circ h$ on V. By an argument of analyticity we have $h^*=\gamma\circ h$ on the entire Ω.

Let us return to $\mathbb{H}^2\times\mathbb{R}$. As $f_w^*=\pm f_w$ in view of (3) we get that $f^*=\pm f+c$ where c is a real constant. At last we obtain $X^*:=(h^*,f^*)=(\gamma\circ h,\pm f+c)$, that is X^* and X differ from an isometry of $\mathbb{H}^2\times\mathbb{R}$.

In the case where $\mathbb{M}=\mathbb{S}^2$ the proof is similar: we use the fact that any harmonic map from Ω into $\mathbb{S}^2$ is the Gauss map of an unique (up to a translation) CMC 1 (possibly branched) immersion into $\mathbb{R}^3$, see [8].

Finally let us consider the case where $\mathbb{M}=\mathbb{R}^2$. Let (g,η) (resp. (g^*,η^*)) be the Weierstrass representation of X (resp. X^*). Therefore X is given by $X=\left(\frac{1}{2}\overline{\int g^{-1}\eta}-\frac{1}{2}\int g\eta,\mathrm{Re}\int\eta\right)$. As $|g^*|=|g|$, we deduce that there exists a real number θ such that $g^*=e^{i\theta}g$. Furthermore we have $\eta=\pm\eta^*$ since $(f_z)^2=(f_z^*)^2=-\phi$. Finally we have $(g^*,\eta^*)=(e^{i\theta}g,\pm\eta)$ and we deduce that X^* differ from X by an isometry of $\mathbb{R}^2\times\mathbb{R}$, this concludes the proof. $\square$

There is also an existence result of minimal immersion into $\mathbb{M}\times\mathbb{R}$ where $\mathbb{M}=\mathbb{H}^2,\mathbb{S}^2$ or $\mathbb{R}^2$.

Theorem 5. *Let $\Omega\subset\mathbb{C}$ be a simply connected domain. Let $ds^2=\lambda^2(w)|\,\mathrm{d}w|^2$ be a conformal metric on Ω and let $Q=\phi(w)\,\mathrm{d}w$ be a holomorphic quadratic differential on Ω with zeros (if any) of even order. Assume that $\mathbb{M}=\mathbb{H}^2,\mathbb{S}^2$ or $\mathbb{R}^2$.*

Then there exists a conformal and minimal immersion $X : \Omega \to \mathbb{M} \times \mathbb{R}$ *such that, setting* $X := (h, f)$*, the Hopf quadratic form of* h *is* Q *(that is* $Q(h) = \phi(w)\,\mathrm{d}w^2$*) and such that the induced metric* ds_X^2 *is*

$$ds_X^2 = ds^2 = \lambda^2(w)|\,\mathrm{d}w|^2$$

if and only if λ *satisfies* $\lambda^2 - 4|\phi| \geq 0$ *and*

$$\Delta\omega = -2K_{\mathbb{M}} \sinh 2\omega|\phi| \tag{16}$$

where $K_{\mathbb{M}}$ *is the (constant) Gauss curvature of* $\mathbb{M}$ *and*

$$\omega := \log \frac{\lambda - \sqrt{\lambda^2 - 4|\phi|}}{2} - \frac{1}{2}\log|\phi|$$

Proof

We first consider the case $K_{\mathbb{M}} = -1$, that is $\mathbb{M} = \mathbb{H}^2$. Let us assume that λ satisfies (16). Consider the 2-form $II := b_{uu}du^2 + 2b_{uv}dudv + b_{vv}dv^2$ on Ω where b_{uu}, b_{uv} and b_{vv} are given by:

$$\begin{cases} b_{uu} + b_{vv} = 2e^{2\omega}|\phi| \\ b_{uu} - b_{vv} = 2\mathrm{Re}(\phi) \\ \qquad\ \ b_{uv} = -\mathrm{Im}(\phi) \end{cases} \tag{17}$$

The Gauss equation for the pair $(e^{2\omega}|\phi||\,\mathrm{d}w|^2, II)$ in $\mathbb{R}^{2,1}$ is:

$$\Delta\omega = -2\sinh(2\omega)|\phi|$$

and then it is satisfied. The Codazzi-Mainardi equations are also satisfied since ϕ is holomorphic. Therefore the fundamental theorem of geometry in $\mathbb{R}^{2,1}$ states that there exists an immersion $\widetilde{X} : \Omega \to \mathbb{R}^{2,1}$ such that the induced metric on Ω is $ds^2_{\widetilde{X}} = e^{2\omega}|\phi||\,\mathrm{d}w|^2$ and the second fundamental form is II. Now the equations (17) show that the immersion has constant mean curvature one.

Up to an isometry of $\mathbb{R}^{2,1}$ we can assume that the Gauss map N of $\widetilde{X}$ takes values in $\mathcal{H}_+$. Therefore $h := \Pi_+ \circ N : \Omega \to \mathbb{H}^2$ is a harmonic mapping such that its Hopf quadratic form is the same as $\widetilde{X}$: $Q(h) = \tilde{\phi}(\widetilde{X})\,\mathrm{d}w^2$, as we have seen in the proof of Theorem 4, see relation (12). By definition we have:

$$\tilde{Q}(\widetilde{X})\,\mathrm{d}w^2 := \frac{1}{2}(b_{uu} - b_{vv} - i2b_{uv})\,\mathrm{d}w^2 = Q.$$

Therefore we obtain $Q(h) = Q$. Moreover we have

$$ds^2_{\widetilde{X}} = ((\sigma \circ h)|h_w|)^2\,|\,\mathrm{d}w|^2$$

and we deduce that $e^{2\omega}|\phi| = ((\sigma \circ h)|h_w|)^2$.

Now we apply Proposition 1 which states that there exists a conformal and minimal immersion $X = (h, f) : \Omega \to \mathbb{H}^2 \times \mathbb{R}$, with induced metric:

$$ds_X^2 = (\sigma^2 \circ h)(|h_w| + |\overline{h}_w|)^2 |\,\mathrm{d}w|^2$$

At last using the fact that $(\sigma \circ h)|\overline{h}_w| = |\phi|/(\sigma \circ h)|h_w|$ we easily compute that:

$$(\sigma^2 \circ h)(|h_w| + |\overline{h}_w|)^2 = \cosh^2 \omega |\phi| |\,\mathrm{d}w|^2 = \lambda^2,$$

that is $ds_X^2 = \lambda^2 |\,\mathrm{d}w|^2$ as desired.

Conversely suppose that such an immersion exists. Then we have by (see (8)):

$$\lambda^2 = 4 \cosh^2 \omega |\phi| |\,\mathrm{d}w|^2.$$

A simple computation shows that we have:

$$\omega = \omega_1 := \frac{1}{2} \log \frac{|\overline{h}_w|}{|h_w|} \quad \text{or} \quad \omega = \omega_2 := \frac{1}{2} \log \frac{|h_w|}{|\overline{h}_w|}.$$

The equation (16) is Böchner formula (10). This concludes the proof in case where $\mathbb{M} = \mathbb{H}^2$.

If $\mathbb{M} = \mathbb{S}^2$ (and then $K_{\mathbb{M}} = 1$), the proof is analogous: we use the fact that for any constant mean curvature one immersion $\widetilde{X} : \Omega \to \mathbb{R}^3$ its Gauss map $N : \Omega \to \mathbb{S}^2$ is harmonic and conversely any harmonic map from Ω into $\mathbb{S}^2$ is the Gauss map of an (possibly branched) immersion into $\mathbb{R}^3$ with constant mean curvature one.

If $K_{\mathbb{M}} = 0$, that is $\mathbb{M} = \mathbb{R}^2$, assume first that ω satisfies (16), that is ω is a harmonic function. As Ω is simply connected, ω is the real part of a holomorphic function $\omega + i\psi$ on Ω. We set:

$$\eta := -2i\sqrt{\phi} \quad \text{and} \quad g := -ie^{\omega + i\psi}.$$

Let $X = (h, f) : \Omega \to \mathbb{R}^2 \times \mathbb{R}$ be the conformal and minimal immersion given by the Weierstrass representation (g, η). $\square$

Observe that in case where $K_{\mathbb{M}} = 0$ the result can be encountered in [4], see the theorem in Section 10.2. We gave the proof for sake of completness. The cases $K_{\mathbb{M}} = 1$ and $K_{\mathbb{M}} = -1$ were proved by B. Daniel in [3] using other methods.

Definition 6. Let M be any riemannian surface. Let $X, X^* : \Omega \to \mathbb{M} \times \mathbb{R}$ be two conformal minimal immersions and let us set $X = (h, f)$ and $X = (h^*, f^*)$.

For any $\theta \in \mathbb{R}$ we say that X and X^* are θ- *associate* (or simply *associate*) if they are isometric immersions and if we have $Q(h^*) = e^{2i\theta} Q(h)$. That is X and

X^* are associate if and only if we have

$$(\sigma\circ h)(|h_w|+|\overline{h}_w|) = (\sigma\circ h^*)(|h^*_w|+|\overline{h^*}_w|) \text{ and } (\sigma\circ h^*)^2 h^*_w \overline{h^*}_w = e^{2i\theta}(\sigma\circ h)^2 h_w \overline{h}_w,$$

where, in a local coordinate (z), the metric on $\mathbb{M}$ is given by $\sigma^2(z)|\,\mathrm{d}\,z|^2$. In case where $\mathbb{M} = \mathbb{R}^2, \mathbb{H}^2$ or $\mathbb{S}^2$, we deduce from the theorem 4 that given a conformal minimal immersion X, the θ-associate minimal immersion is uniquely determined up to an isometry of $\mathbb{M}\times\mathbb{R}$. Furthermore if $\theta = \pi/2$ we say that X and X^* are *conjugate*.

Remark 7. *Two isometric immersions X and X^θ are associate up to an isometry if $\eta^\theta = e^{i\theta}\eta$ and by (8) $|g^\theta| + |g^\theta|^{-1} = |g| + |g|^{-1}$ (or equivalently $\cosh\omega^\theta = \cosh\omega$). Then $\omega^\theta = \omega$ or $\omega^\theta = -\omega$. In particular X and X^θ are associate if and only if $N_3(X) = N_3(X^*)$ or $N_3(X) = -N_3(X^*)$ (recall that $N_3(X) = \tanh\omega$) and $\eta^\theta = e^{i\theta}\eta$.*

In fact B. Daniel proved that the associate family always exists in $\mathbb{H}^2\times\mathbb{R}$ and $\mathbb{S}^2\times\mathbb{R}$, see [3]. In this situation he gave an alternative definition of associate and conjugate isometric immersions which turn to be equivalent to our definition. We are going to give another proof of the existence of the associate family.

***Corollary* 8.** *Let $X := (h, f) : \Omega \to \mathbb{M}\times\mathbb{R}$ be any conformal and minimal immersion where $M = \mathbb{H}^2$, $\mathbb{S}^2$ or $\mathbb{R}^2$. Then for any $\theta \in \mathbb{R}$ there exists a $\theta -$ associate immersion $X_\theta := (h_\theta, f_\theta) : \Omega \to \mathbb{M}\times\mathbb{R}$. Furthermore $X_0 = X$ and X_θ is unique up to isometry of $\mathbb{M}\times\mathbb{R}$.*

Proof

Let us set $Q(h) = \phi(w)dw^2$ and let ds^2_X be the conformal metric induced on Ω by X. We deduce from Theorem 5 that the pair (ds^2_X, ϕ) satisfies the condition (16). Therefore for any $\theta \in \mathbb{R}$ the pair $(ds^2_X, e^{2i\theta}\phi)$ also satisfies condition (16). Finally we infer with Theorem 5 that there exists a θ-associate immersion, which concludes the proof. □

4. Minimal vertical graph

In this section we study geometric properties of minimal graph and their associate family. Recall from the introduction that we introduce some "Weierstrass" data for minimal surfaces (g, η) with $g = -ie^{\omega+i\psi}$ and $\eta = -2i\sqrt{\phi}$. When X is a minimal surface of $\mathbb{R}^3$, then $\omega + i\psi$ is meromorphic. In the other case ω satisfy the sinh-Gordon equation (11). In the following Lemma, we determine how the function $\omega + i\psi$ deviate from to be meromorphic. We express same expression for associate family. In this case (remark 7), up to an isometry we have $g^\theta = -ie^{\omega+i\psi^\theta}$ and $\eta^\theta = e^{i\theta}\eta$ ($\omega^\theta = \omega$). Then we have:

Lemma 9. *We consider a harmonic map $h : \Omega \to (U, \sigma^2 |\,\mathrm{d}z|^2)$ with holomorphic quadratic Hopf differential $Q = \phi(w)(\mathrm{d}w)^2$ with zeros (if any) of even order and coefficient of dilatation $a(z) = e^{-2(\omega + i\psi)}$. Thus we can define $\sqrt{\phi} = |\phi|^{1/2} e^{i\beta}$ and we identify σ with $\sigma \circ h$. Then*

$$(\omega + i\psi)_{\bar{w}} = |\phi|^{1/2} e^{-i\beta} \left(\sinh \omega \langle \frac{\nabla \log \sigma}{\sigma}, e^{i\psi} \rangle + i \cosh \omega \langle \frac{\nabla \log \sigma}{\sigma}, i e^{i\psi} \rangle \right) \tag{18}$$

***Corollary* 10.** *If $X = (h, f)$ is a minimal surface and $X^\theta = (h^\theta, \eta^\theta)$ is the associate family of X define in the Definition 6, we can define the map $\omega^\theta + i\psi^\theta$ and with the notation $\sigma^\theta = \sigma \circ h^\theta$ we have $\omega^\theta = \omega$ and*

$$(\omega + i\psi^\theta)_{\bar{w}} = |\phi|^{1/2} e^{-i(\beta + \theta)} \left(\sinh \omega \langle \frac{\nabla \log \sigma^\theta}{\sigma^\theta}, e^{i\psi^\theta} \rangle + i \cosh \omega \langle \frac{\nabla \log \sigma^\theta}{\sigma^\theta}, i e^{i\psi^\theta} \rangle \right)$$

Proof

We compute ψ_u as a function of ω_v and ψ_v as a function of ω_u. In complex coordinates w, using (6), (9) and assuming $\eta = -2i\sqrt{\phi}$ we derive:

$$h_w = \frac{\sqrt{\phi} e^{\omega + i\psi}}{\sigma} \text{ and } h_{\bar{w}} = \frac{\overline{\sqrt{\phi}} e^{-\omega + i\psi}}{\sigma}$$

while

$$h^\theta_w = \frac{e^{i\theta} \sqrt{\phi} e^{\omega + i\psi^\theta}}{\sigma^\theta} \text{ and } h^\theta_{\bar{w}} = \frac{e^{-i\theta} \overline{\sqrt{\phi}} e^{-\omega + i\psi^\theta}}{\sigma^\theta}.$$

Inserting these expressions in the harmonic equation (1) we obtain:

$$(\omega + i\psi)_{\bar{w}} = -\sigma \left(\frac{1}{\sigma} \right)_{\bar{z}} - 2(\log \sigma)_u h_{\bar{w}}$$

$$(\omega + i\psi^\theta)_{\bar{w}} = -\sigma^\theta \left(\frac{1}{\sigma^\theta} \right)_{\bar{w}} - 2(\log \sigma^\theta)_u h_{\bar{w}}.$$

Now note that

$$\begin{aligned} -\sigma \left(\frac{1}{\sigma} \right)_{\bar{w}} &= (\log \sigma)_{\bar{w}} \\ &= \left((\log \sigma)_z h_{\bar{w}} + (\log \sigma)_{\bar{z}} \bar{h}_{\bar{w}} \right) \end{aligned}$$

where $2(\log \sigma)_z = (\log \sigma)_x - i(\log \sigma)_y$ and $\bar{h}_{\bar{z}} = \overline{h_z}$. Collecting these equations we obtain:

$$(\omega + i\psi)_{\bar{w}} = (\log \sigma)_{\bar{z}} \bar{h}_{\bar{w}} - (\log \sigma)_z h_{\bar{w}}.$$

which is

$$(\omega + i\psi)_{\bar{w}} = \frac{|\phi|^{1/2} e^{-i\beta}}{\sigma} (\sinh\omega\,(\cos\psi(\log\sigma)_x + \sin\psi(\log\sigma)_y)$$
$$+ i\cosh\omega\,(\cos\psi(\log\sigma)_y - \sin\psi(\log\sigma)_x))$$

Since X^θ is isometric to X, we have $\omega^\theta = \omega$ and the same equation applied to h^θ yields,

$$(\omega + i\psi^\theta)_{\bar{w}} = (\log\sigma^\theta)_{\bar{z}} \bar{h}^\theta_{\bar{w}} - (\log\sigma^\theta)_z h^\theta_{\bar{w}}.$$

gives

$$(\omega + i\psi^\theta)_{\bar{w}} = \frac{|\phi|^{1/2} e^{-i(\beta+\theta)}}{\sigma^\theta} \left(\sinh\omega\left(\cos\psi^\theta(\log\sigma^\theta)_y + \sin\psi^\theta(\log\sigma^\theta)_y\right)\right.$$
$$\left. - i\cosh\omega\left(\cos\psi^\theta(\log\sigma^\theta)_y - \sin\psi^\theta(\log\sigma^\theta)_y\right)\right)$$

□

We consider the projection $F : \mathbb{M} \times \mathbb{R} \longrightarrow \mathbb{M} \times \{0\}$, thus $F \circ X = h$. Now let us consider a curve $\gamma : [0, l] \longrightarrow \Omega \subset \mathbb{C}$ parametrized by arclenght and $\gamma'(t) = e^{i\alpha(t)}$ in $\Omega \subset \mathbb{C}$. We will compute in the following what are the curvature k in $\mathbb{M}$ of the planar curves $F \circ X(\gamma) = h(\gamma)$ and $F \circ X^\theta(\gamma) = h^\theta(\gamma)$. A such computation appears in [7] in the particular case where $\alpha = 0$ and $\alpha = \pi/2$:

Proposition 11. *If we consider a curve γ in Ω and the image $h(\gamma))$ and $h^\theta(\gamma)$ in M, then the curvature are*

$$k(h(\gamma)) = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v + G_t}{2|\phi|^{1/2} R} \tag{19}$$

$$k(h^\theta(\gamma)) = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v + G^\theta_t}{2|\phi|^{1/2} R^\theta} \tag{20}$$

where

$$Re^{iG} = \cos(\alpha + \beta)\cosh\omega + i\sin(\alpha + \beta)\sinh\omega$$

$$R^\theta e^{iG^\theta} = \cos(\alpha + \beta + \theta)\cosh\omega + i\sin(\alpha + \beta + \theta)\sinh\omega$$

Proof

We apply the formula (6) with $g = -ie^{\omega + i\psi}$, $\eta = -2i\sqrt{\phi}dz$ for X and $g^\theta = -ie^{\omega + i\psi^\theta}$ and $\eta^\theta = e^{i\theta}\eta = -2ie^{i\theta}\sqrt{\phi}dz$. Let us recall $\sqrt{\phi} = |\phi|^{1/2} e^{i\beta}$.

$$\begin{aligned}
\frac{dh(\gamma)}{dt} &= \frac{2|\phi|^{1/2}}{\sigma}\cosh(\omega + i\alpha + i\beta)e^{i\psi} \\
&= \frac{2|\phi|^{1/2}}{\sigma}(\cos(\alpha+\beta)\cosh\omega + i\sin(\alpha+\beta)\sinh\omega)e^{i\psi} \\
\frac{dh(\gamma)}{dt} &= \frac{2|\phi|^{1/2}}{\sigma}Re^{i(\psi+G)}
\end{aligned}$$

$$\begin{aligned}
\frac{dh^\theta(\gamma)}{dt} &= \frac{2|\phi|^{1/2}}{\sigma}\cosh(\omega + i\beta + i\alpha + i\theta)e^{i\psi^\theta} \\
&= \frac{2|\phi|^{1/2}}{\sigma}(\cos(\alpha+\beta+\theta)\cosh\omega + i\sin(\alpha+\beta+\theta)\sinh\omega)e^{i\psi^\theta} \\
\frac{dh^\theta}{dt} &= \frac{2|\phi|^{1/2}}{\sigma}Re^{i(\psi^\theta+G_\theta)}
\end{aligned}$$

If k is the curvature of a curve γ in $(U, \sigma^2(z)|\,\mathrm{d}z|^2)$ and k_e is the Euclidean curvature in $(U, |dz|^2)$, we get by conformal change of the metric:

$$k = \frac{k_e}{\sigma} - \frac{\langle \nabla\sigma, n\rangle}{\sigma^2}$$

where n is the Euclidean normal to the curve γ such that (γ', n) is positively oriented. If s denotes the arclength of $h(\gamma)$ and s^θ the arclength of $h^\theta(\gamma)$, we have

$$k_e(h(\gamma)) = \psi_s + G_s = \frac{\sigma}{2|\phi|^{1/2}R}(\cos\alpha\psi_u + \sin\alpha\psi_v) + G_s \tag{21}$$

We consider the euclidean normal of $h(\gamma)$ (resp $h^\theta(\gamma)$) given by

$$\begin{aligned}
n &= (-\sin(\alpha+\beta)\sinh\omega + i\cos(\alpha+\beta)\cosh\omega)\frac{e^{i\psi}}{R} \\
n^\theta &= (-\sin(\alpha+\beta+\theta)\sinh\omega + i\cos(\alpha+\beta+\theta)\cosh\omega)\frac{e^{i\psi^\theta}}{R^\theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\langle\nabla\sigma, n\rangle}{\sigma^2} = \frac{\langle\nabla\log\sigma, n\rangle}{\sigma} &= -\sin(\alpha+\beta)\frac{\sinh\omega}{R}\langle\frac{\nabla\log\sigma}{\sigma}, e^{i\psi}\rangle \\
&\quad + \cos(\alpha+\beta)\frac{\cosh\omega}{R}\langle\frac{\nabla\log\sigma}{\sigma}, ie^{i\psi}\rangle
\end{aligned}$$

Using the Lemma 9, we express ψ_u in terms of ω_v, and ψ_v in terms of ω_u by using formula (18) in (21), we deduce

$$\frac{\psi_s}{\sigma} - \frac{\langle \nabla\sigma, n \rangle}{\sigma^2} = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v}{2|\phi|^{1/2}R}$$

The same computation with X^θ yields

$$\frac{\psi^\theta_{s^\theta}}{\sigma^\theta} - \frac{\langle \nabla\sigma^\theta, n \rangle}{(\sigma^\theta)^2} = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v}{2|\phi|^{1/2}R^\theta}$$

This concludes the proof of the proposition since $G_s = \frac{\sigma}{2|\phi|^{1/2}R} G_t$. □

Now we prove the generalization of the Krust's theorem for minimal vertical graph and associate family surfaces. Let $U \subset \mathbb{M}$ be an open set and $f(z)$ a smooth function on U. We say that G is a vertical graph in $\mathbb{M} \times \mathbb{R}$ if $G = \{(z,t) \in \mathbb{M} \times \mathbb{R}; t = f(z),\ z \in U\}$. The graph is an entire vertical graph if $U = \mathbb{M}$.

Theorem 12. *If we consider a minimal graph $X(\Omega)$ on a convex domain $h(\Omega)$ in $\mathbb{M}$, then the associate surface $X^\theta(\Omega)$ is a graph under the assumption that the curvature $K_{\mathbb{M}} \leq 0$.*

When $K_{\mathbb{M}} \equiv 0$ it is a result of Romain Krust (see [6], page 188 and application therein).

Proof

The proof is a direct application of Gauss-Bonnet theorem with the fact that ω is without zero (X is a vertical graph). If we consider a smooth piece of embedded curve Γ in $\mathbb{M}$ with end points p_1 and p_2, then if $p_1 = p_2$, Γ is enclosing a region A and:

$$\int_A K_{\mathbb{M}} dV_\sigma + \int_\Gamma k(s)ds + \alpha = 2\pi$$

where α is the exterior angle at $p_1 = p_2$. Since Γ is embedded we have $\alpha \leq \pi$. The Gauss Bonnet formula above gives us that in the case where $K_{\mathbb{M}} \leq 0$:

$$\pi \geq \alpha \geq 2\pi - \int_\Gamma k ds.$$

Now, if we assume that $X^\theta(\Omega)$ is not a graph, there exist two points p_1 and p_2 two points of Ω with $h^\theta(p_1) = h^\theta(p_2)$. Since $h(\Omega)$ is convex, there is a geodesic in $h(\Omega)$ which can be lift by a path γ in Ω. In summary, we assume that the curve $\gamma(t)$, $t \in [0, l]$ is parametrized by euclidean arclength, $\gamma'(t) = e^{i\alpha}$, $h(\gamma)$ is a piece of a geodesic of $\mathbb{M}$, p_1 and p_2 are the end points of γ and $h^\theta(p_1) = h^\theta(p_2)$. We assume that $h^\theta(\gamma)$ is a closed embedded curve. If not, we can consider a subarc of γ with end points p'_1 and p'_2, with an image by h^θ smooth, embedded

and $h^\theta(p'_1) = h^\theta(p'_2)$. In the case where this subarc embedded doesn't exist, then it is meaning that all points are double, like a path where we go and back after an interior end point q. At q, $h^\theta(\gamma)$ is not immersed. The derivative is nul and then the tangent plane of X^θ is vertical. Then ω would have an interior zero (a contradiction with the vertical graph assumption).

We will apply the formula of Gauss-Bonnet and we will prove that $\int_{h^\theta(\gamma)} kds^* < \pi$ under the hypothesis that $h(\gamma)$ is a geodesic. It will provide a contradiction with $\alpha \leq \pi$ and then the horizontal curve cannot be closed and embedded.

If $h(\gamma)$ is a geodesic, then by the formula (19) of the previous proposition 11, we have $\sin\alpha\omega_u - \cos\alpha\omega_v + G_t = 0$. Thus

$$k(h^\theta(\gamma)) = \frac{G_t^\theta - G_t}{2|\phi|^{1/2} R^\theta}$$

Since $ds^* = 2|\phi|^{1/2} R^\theta dt$, we have

$$\int_{h^\theta(\gamma)} kds^* = (G^\theta(l) - G(l)) - (G^\theta(0) - G(0))$$

Now, we remark with a direct computation of the real and imaginary part of

$$z = \frac{R^\theta}{R} e^{i(G^\theta - G)} = \frac{\cos(\alpha+\beta+\theta)\cosh\omega + i\sin(\alpha+\beta+\theta)\sinh\omega}{\cos(\alpha+\beta)\cosh\omega + i\sin(\alpha+\beta)\sinh\omega}$$

that

$$\tan(G^\theta(t) - G(t)) = \frac{\sinh(2\omega)\sin\theta}{2\cos\theta R^2 - \sin\theta\sin 2(\alpha+\beta)}$$

.

Since X is a graph ω is without interior zero, and then $\sinh(2\omega)\sin\theta$ cannot be zero for $\theta \in (0, \pi/2]$. It implies that $G^\theta(t) - G(t) \in [0, \pi]$. $\square$

The example 13 below prove that the conjugate surface of an entire graph in $\mathbb{H}^2 \times \mathbb{R}$ is a graph (by our theorem 12) but it is not necessary entire. In this direction we give the following criterion in $\mathbb{H}^2 \times \mathbb{R}$:

Theorem 13. *Let $X : \mathbb{D}^2 \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ be an entire vertical graph on $\mathbb{H}^2$. Then on any divergent path γ of finite euclidean lenght in $\mathbb{D}$*

$$\int_\gamma |\phi|^{1/2} dt < \infty$$

then the conjugate graph X^* is an entire graph.

Proof

Recall that $f+if^* = -2i\int^z \sqrt{\phi}$ is holomorphic. We consider $\gamma(t)$ a divergent path in $\mathbb{D}^2$ and $X(\gamma) = \Gamma$ its image in the graph (recall that $\gamma'(t) = e^{i\alpha})$). Since X is a proper map, the length of Γ is infinite in X and

$$\ell(\Gamma) = \int_\gamma 2\cosh\omega|\phi|^{1/2}dt = \infty \tag{22}$$

Now we prove that the length of $h^*\circ\gamma$ is infinite which prove the theorem. If X^* is not entire, one can find a diverging curve in $\mathbb{D}^2$ with $h^*\circ\gamma$ of finite length. Now let us compute

$$\ell(h^*\circ\gamma) = \int_\gamma 2|\phi|^{1/2}R^*dt$$

where $R^{*2} = \sin^2(\alpha+\beta)\cosh^2\omega + \cos^2(\alpha+\beta)\sinh^2\omega$ (recall $R^* = R^{\pi/2}$). Now we remark that

$$R^{*2} = \cosh^2\omega - \cos^2(\alpha+\beta).$$

Then, using (22), and the hypothesis we have

$$\ell(h^*\circ\gamma) = \ell(\Gamma) - \int_\gamma |\cos(\alpha+\beta)||\phi|^{1/2}dt = \infty.$$

$\square$

5. Examples

Example 14. Let us consider the *Scherk type surface* in $\mathbb{H}^2\times\mathbb{R}$ invariant by hyperbolic translations. It is a complete minimal graph over a non-bounded domain in $\mathbb{H}^2$ defined by a complete geodesic γ in $\mathbb{H}^2\times\{0\}$. The graph takes $\pm\infty$ value on γ and 0 value on the asymptotic boundary. In the half plane model of $\mathbb{H}^2 = \{(x,y)\in\mathbb{R}^2,\ y>0\}$, there is a nice formula for the graph:

$$t = \ln\left(\frac{\sqrt{x^2+y^2}+y}{x}\right), \qquad y>0,\ x>0$$

The Scherk's conjugate minimal surface in $\mathbb{H}^2\times\mathbb{R}$ (see Theorem 4.2 of [12]) is given by the equation: $t = x$. It is invariant by parabolic screw-motions. It is a whole graph over $\mathbb{H}^2$. The second and third authors proved that in $\mathbb{H}^2\times\mathbb{R}$, a catenoid is conjugate to a helicoid of pitch $\ell<1$, see [14]. Surprisingly, a helicoid of pitch $\ell = 1$ is conjugate to a surface invariant by parabolic translations, see [3] or [12]. Furthermore any helicoid with pitch $\ell>1$ is conjugate to a minimal surface invariant by hyperbolic translations, see [12].

Remark 15. *Assume that* $\mathbb{M} = \mathbb{R}^2$ *and let us consider* $X, X^* : \Omega \to \mathbb{R}^2 \times \mathbb{R}$ *two conformal minimal immersions. Let* (g, η) *(resp.* (g^*, η^*)*) be the Weierstrass representation of* X *(resp.* X^**). We know that* X *and* X^* *are associate in the usual meaning, that is in the Euclidean space* $\mathbb{R}^3$*, if and only if* $g^* = g$ *and* $\eta^* = e^{i\theta}\eta$ *for a real number* θ*. Let us set* $X = (h, f)$ *and* $X^* = (h^*, f^*)$*. Since* $Q(h) = -\frac{(\eta)^2}{4}$ *we see that if* X *and* X^* *are associate in the usual meaning then there are associate in the meaning of definition 6. Conversely, assume that* X *and* X^* *are associate in the sense of definition 6, so we have* $\eta^* = \pm\eta$ *and* $|g^*| = |g|$ *or* $|g^*| = 1/|g|$*. Therefore there exists an isometry* Γ *of* $\mathbb{R}^3$ *such that* X *and* $\Gamma \circ X^*$ *are associate in the usual meaning.*

For example the Weierstrass representations (g, η) *and* $(e^{i\theta}g, e^{i\theta}\eta)$ *for* $\theta \neq 2k\pi,\ k \in \mathbb{Z}$*, are associate in the sense of definition 6 but are not in the usual sense.*

Therefore, in $\mathbb{R}^2 \times \mathbb{R}$ *the two notions of associate minimal immersions are only equivalent up to an isometry of* $\mathbb{R}^2 \times \mathbb{R}$*.*

It is known that any two isometric conformal minimal immersions in $\mathbb{R}^3$ are associate up to an isometry. Also, it is shown in [14] that any two isometric screw motion minimal complete immersions in $\mathbb{H}^2 \times \mathbb{R}$ are associate. The following example shows that this is no longer true for any isometric immersions in $\mathbb{H}^2 \times \mathbb{R}$.

Example 16. It is given in [14] an example of complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with intrinsic curvature constant and equals to -1, $K \equiv -1$. Namely in (48) of Corollary 21 of [14] we set, keeping the notations, $H = 0,\ l = m = 1$, d is any positive real number, $d > 0$. Therefore $U(s) = \sqrt{1 + d^2}\cosh(s)$, $s \in \mathbb{R}$. Consequently from Theorem 19 in [14] we obtain (see (33), (36) and (37))

$$\begin{aligned}
\rho(s) &= \operatorname{arcosh}(\sqrt{1 + d^2}\cosh s) \\
\lambda(\rho(s)) &= d\int \frac{U(s)}{U^2(s) - 1}ds \\
\varphi(s, \tau) &= \tau - d\int \frac{1}{U(s)(U^2(s) - 1)}ds
\end{aligned}$$

Now let us consider the map $T : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ defined for every $(s, \tau) \in \mathbb{R}^2$ by

$$T(s, \tau) = (\tanh(\rho(s)/2)\cos\varphi(s, \tau), \tanh(\rho(s)/2)\sin\varphi(s, \tau), \lambda(\rho(s)) + \varphi(s, \tau))\,,$$

It is shown that T is a regular minimal *embedding* with induced metric

$$ds_T^2 = ds^2 + U^2(s)d\tau^2 = ds^2 + (1 + d^2)\cosh^2(s)d\tau^2.$$

A straightforward computation shows that the intrinsic curvature is given by $K = -U''/U$. Therefore we have $K \equiv -1$. By construction the surface $T(\mathbb{R}^2)$ is invariant by screw motions. The immersion T is not conformal but setting $r := \int (1/U(s))ds$ the new coordinates (r, τ) are conformal, that is the immersion $\widetilde{T}(r, \tau) := T(s, \tau)$ is conformal. Thus the surface $T(\mathbb{R}^2)$ is isometric to the hyperbolic plane $(\mathbb{D}, \sigma^2(z)|\,dz|^2)$, that is there exists a conformal minimal immersion $X : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{R}$ such that the induced metric on $\mathbb{D}$ is the hyperbolic one and $X(\mathbb{D}) = T(\mathbb{R}^2)$. Clearly the canonical immersion $Y : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{R}$ defined by $Y(z) = (z, 0)$ is isometric to X. According to the remark 7, we deduce that X and Y are not associate since the third component of the Gauss map of X never is equal to ± 1 as it is the case for Y.

Let us observe that in [7] it can be found other examples of complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with intrinsic curvature equal to -1.

Remark 17. *The second author has constructed in* [12] *new families of complete minimal immersions in* $\mathbb{H}^2 \times \mathbb{R}$ *invariant by parabolic or hyperbolic screw motions. It is shown (see Theorem 4.1) that any two minimal isometric parabolic screw motion immersions into* $\mathbb{H}^2 \times \mathbb{R}$ *are associate. However this is no longer true for hyperbolic screw motion immersions: there exist isometric minimal hyperbolic screw motion immersions into* $\mathbb{H}^2 \times \mathbb{R}$ *which are not associate, see Theorem 4.2.*

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